

Matching Spherical Dust Solutions to Construct Cosmological Models

D. R. Matravers* and N. P. Humphreys†

*Relativity and Cosmology group, School of Computer Science and Mathematics, University of Portsmouth,
Portsmouth PO1 2EG, Britain*

Conditions for smooth cosmological models are set out and applied to inhomogeneous spherically symmetric models constructed by matching together different Lemaitre-Tolman-Bondi solutions to the Einstein field equations. As an illustration the methods are applied to a collapsing dust sphere in a curved background. This describes a region which expands and then collapses to form a black hole in an Einstein de Sitter background. We show that in all such models if there is no vacuum region then the singularity must go on accreting matter for an infinite LTB time.

I. INTRODUCTION

Recently we and others [1,2] have had cause to use matched spherically symmetric solutions of the field equations to model fractal [3] distributions of matter in cosmological situations. Single spherically symmetric models (see [4] for a recent example) and matched ones have long been used in cosmology (see [5] for an extensive and comprehensive review). They are remarkably rich in structure but there are many subtleties in their application to cosmology [6–8]. The purpose of this paper is to clarify and unify existing results on the construction of *smooth* models. The topic has important implications for the modelling of structures like voids or the formation of black holes in curved backgrounds. The new results include matching between exact solutions in the Kantowski-Sachs family and the Lemaitre-Tolman-Bondi (LTB) solutions and to the occurrence of centres. It is not always realised that a spherically symmetric model need not possess a centre [9]. Here an extended list of the possible types of centre is given. Also all possible regular composite dust models are given and they are classified into four classes according to the number of centres they allow. One of the classes in which the spatial sections have the topology of a 3-torus appears to be new.

This work complements that by Lake [10] and by Fayos et al. [11]. The work by Lake is more theoretical. That by Fayos, Senovilla and Torres is more general and very interesting geometrically but we find it less easy to see some subtle problems such as the one we illustrate in the final section. It has long been accepted that a black hole in an Einstein-de Sitter model can be constructed using a Schwarzschild interface. What had not been shown but is demonstrated here is how a black hole evolves from an expanding region in an Einstein de Sitter background. The model is a modification of one produced by Papapetrou [20] which is non-physical because the singularity takes an infinite LTB time to be completed. In section 7 it is shown that this infinite time is an inevitable result of the matching conditions if no vacuum region is included, a result which appears to be not widely known.

II. SPHERICAL DUST SOLUTIONS

In comoving coordinates $x^a = \{t, r, \theta, \phi\}$ the spherically symmetric metric can be written [7]

$$ds^2 = -dt^2 + X^2(r, t)dr^2 + R^2(r, t)d\Omega^2, \quad (1)$$

and the dust 4-velocity as $u^a = \delta_t^a$. Here $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$ and $R \geq 0$ with $R = 0$ only at a centre. The coordinate ranges are; $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $t > T(r)$ (see the solutions below for a definition of T). The dust energy momentum tensor is $T_b^a = \mu u^a u_b$, where μ is the proper matter energy density.

In order for the Einstein tensor to be well defined we require that

$$X \neq 0 \neq R; R \text{ is } C^2 \text{ in } r \text{ and } t; X \text{ is } C^2 \text{ in } t \text{ and } C^1 \text{ in } r.$$

We call such behaviour regular. Regions of regularity may be joined together to form composite space-times in which the differentiability conditions hold only piece-wise. More details are given below when we discuss the matching. To set the notation and for later use we first write down the first integrals and write out the solutions [12,13,7]. We will

*david.matravers@port.ac.uk.

†humphrn@uk.ibm.com

call the solutions LTB models and refer to the coordinates as LTB coordinates. We use units such that $G = c = 1$ and the notation, overdot for $u^a \partial_a = \partial/\partial t$ and prime for $\partial/\partial r$. If $R' \neq 0$, then the Einstein field equations reduce to

$$X = \frac{R'}{\sqrt{1 + E(r)}}, \quad (2)$$

where $E(r)$ is an arbitrary function of integration, and

$$\dot{R}^2 = \frac{2M(r)}{R} + E, \quad (3)$$

with $M(r)$ a second function of integration. The proper density is given by

$$\mu = \frac{M'}{4\pi R' R^2}. \quad (4)$$

The equations (2) - (4) have five solutions:

$$\begin{aligned} [\mathbf{s1}] \text{ for } \{E = M = 0\}, \quad & R = -T(r), \\ [\mathbf{s2}] \text{ for } \{E > 0, M = 0\}, \quad & R = \sqrt{E}(\epsilon t - T), \\ [\mathbf{s3}] \text{ for } \{E = 0, M > 0\}, \quad & R = (9M/2)^{1/3}(\epsilon t - T)^{2/3}, \\ [\mathbf{s4}] \text{ for } \{E > 0, M > 0\}, \quad & R = \frac{M}{E}(\cosh \eta - 1), \\ & \sinh \eta - \eta = (\epsilon t - T)E^{3/2}M^{-1}, \\ & 0 < \eta < \infty, \\ [\mathbf{s5}] \text{ for } \{E < 0, M > 0\}, \quad & R = M(\cos \eta - 1)E^{-1}, \\ & \eta - \sin \eta = (\epsilon t - T)|E|^{3/2}M^{-1}, \\ & 0 < \eta < 2\pi. \end{aligned}$$

Here, $T(r)$ is a further function of integration and $\epsilon = \pm 1$. The solutions **s1** and **s2** are locally Minkowskian. The surfaces $\{\epsilon t - T = 0\}$ are spacelike and singular. Following Bondi [7], M is the relativistic generalisation of Newtonian mass and $\frac{1}{2}E$ is the total energy.

The case $R' = 0$ leads to an inhomogeneous generalisation of the Kantowski-Sachs metric [15] [16] which does not have centres of symmetry in the hypersurfaces $t = \text{constant}$. It is given in our notation by

[**s6**]

$$R = \bar{M}(1 - \cos \eta), \quad (5)$$

$$\eta - \sin \eta = \bar{M}^{-1}(\epsilon t - T), \quad (6)$$

$$X = A(r) \frac{\sin \eta}{1 - \cos \eta} + B(r) \left[1 - \frac{\eta \sin \eta}{2(1 - \cos \eta)} \right], \quad (7)$$

where A and B are integration functions, \bar{M} is a constant, and $0 < \eta < 2\pi$. We label this solution **s6**.

III. JUNCTION CONDITIONS

Our aim is to establish a class of cosmological models which do not have shell-crossing or surface density layers, but which can be constructed by matching together different LTB solutions from the set **s1** to **s6**, so we impose the Darmois conditions [14] on the junctions. For a comoving space-like junction $r = \text{constant}$ and dust, these require that the first and second fundamental forms are continuous across the junction (interface). The unit normal is given by $n_a = |X| \delta_a^r$, and the first fundamental form is the metric intrinsic to the interface, i.e.,

$$h_{ab} = g_{ab} - n_a n_b \quad (8)$$

$$= \text{diag}(-1, 0, R^2, R^2 \sin^2 \theta), \quad (9)$$

and the second fundamental form is the extrinsic curvature which is given by

$$K_{ab} = h_a^c h_b^d \nabla_d n_c \quad (10)$$

$$= \text{diag} \left(0, 0, \frac{RR'}{|X|}, \frac{RR'}{|X|} \sin^2 \theta \right). \quad (11)$$

From these it follows that the necessary and sufficient conditions for matching are that

$$R \quad \text{is continuous in } r, \quad (12)$$

$$\frac{R'}{|X|} \quad \text{is continuous in } r. \quad (13)$$

The nature of the problem changes in the non-comoving case [16] [17]. The conditions at the junction must be satisfied through some range of values of r . Explicitly, if the spacelike non-comoving boundary surface is given by

$$r - g(t) = \text{constant}, \quad (14)$$

where for convenience we choose P such that

$$\frac{dg}{dt} = \frac{P}{|X|\sqrt{P^2 - 1}}, \quad (15)$$

then the unit normal to the surface is given by

$$n^a = \left(P, \frac{1}{|X|\sqrt{P^2 - 1}}, 0, 0 \right) \quad (16)$$

and $n^a n_a = -1$. The unit tangent to the surface with $\theta = \phi = \text{constant}$ is

$$m^a = \left(\sqrt{P^2 - 1}, \frac{P}{|X|}, 0, 0 \right). \quad (17)$$

The intrinsic curvature of the surface is given by

$$\hat{h}_{ab} = \begin{pmatrix} P^2 - 1 & -P|X|\sqrt{P^2 - 1} & 0 & 0 \\ -P|X|\sqrt{P^2 - 1} & P^2 X^2 & 0 & 0 \\ 0 & 0 & R^2 & R^2 \sin^2 \theta \end{pmatrix}, \quad (18)$$

and the extrinsic curvature is

$$\hat{K}_{ab} = \begin{pmatrix} (P^2 - 1)F_1 & -F_1|X|P\sqrt{P^2 - 1} & 0 & 0 \\ -F_1|X|P\sqrt{P^2 - 1} & P^2 X^2 F_1 & 0 & 0 \\ 0 & 0 & F_2 & 0 \\ 0 & 0 & 0 & F_2 \sin^2 \theta \end{pmatrix}, \quad (19)$$

where $F_1 = \frac{dP}{dt} + P\frac{\dot{X}}{X}$ and $F_2 = PR\frac{dR}{dt} - \frac{RR'}{|X|\sqrt{P^2 - 1}}$, and all the ordinary derivatives are taken along the paths $\{\theta, \phi\} = \text{constant}$ in the tangent space to the hypersurface. On the hypersurface, one coordinate is surplus because of (14).

We will not deal with non-comoving junctions further here except to mention two points. First that at the junction at least one of E , M and T must be constant for a range values of r . This follows from the fact that they have to be continuous by the Darmois conditions and at least one of them has at most one value in common (or as a limit for **s6**) for allowed¹ matches between the solutions **s1** to **s6**. Thus in the non-comoving case there will be an interface

¹see later

region. Second, that Krasiński [18] suggests that non-comoving boundaries could find an application in the formation of structure. The idea is that it may be possible to use them to allow incoming matter to augment the condensations discussed by Bonnor [19].

From here on we restrict attention to junctions where there are no surface layers or interface region and so they are necessarily comoving and the metrics on both sides of the junction are determined by the integration functions $E(r)$, $M(r)$ and $T(r)$ (and $A(r)$ and $B(r)$ in regions where $E(r) = -1$). For the spherically symmetric dust models the junction conditions (12) imply that

$$M(r)(\geq 0), E(r)(\geq -1) \text{ and } T(r) \text{ are continuous.} \quad (20)$$

IV. REGULARITY REQUIREMENTS

In this section the physical requirements to be imposed on the metrics are made more explicit and justified. It follows from the conditions on the metric and the matching conditions across spacelike surfaces that the metrics **s1** to **s6** are at least C^2 in t . Thus we are only concerned with the behaviour with respect to the radial coordinate r . We start by requiring that $\lim_{\pm} X \neq 0$ everywhere including across the junctions. This condition is important because it enables us to express the continuity properties of physical quantities unambiguously through their differentiability in r , and prevents shell crossing. To ensure that there is not a curvature singularity as $R \rightarrow 0$, we require that M/R^3 is finite everywhere except trivially at the spacelike singularity $\tau \rightarrow 0$. Finally we require that at a centre the shear

$$\sigma_a^b \equiv \frac{1}{3} \left[\frac{\dot{X}}{X} - \frac{\dot{R}}{R} \right] \times \text{diag}(0, 2, -1, -1)$$

go to zero to maintain spherical symmetry. If it is not zero, then the eigendirections of the shear tensor will violate spherical symmetry at the centre.

In summary, we require the following properties to hold in our matched spacetimes:

R1 The junction conditions (12,13) hold.

R2 $\lim_{\pm} X \neq 0$.

R3 The shear tends to zero whenever $R \rightarrow 0$, i.e. at a centre.

R4 M/R^3 remains finite as $R \rightarrow 0$ (except trivially at the spacelike singularity $\tau \rightarrow 0$).

R5 The metrics **s1** to **s6** are regular within their domains, i.e. between junctions.

We will now describe the implications of these conditions within the domains of metrics. It follows from the field equation (2) that

$$R' \text{ can change sign only at values of } r \text{ for which } E(r) = -1, \quad (21)$$

i.e., in **s5**. For **s1** to **s4**, $R' \geq 0$ or $R' \leq 0$ throughout. For metrics **s1** to **s4**, differentiation of the exact solutions for R , and examination of the asymptotic behaviour for large and small t , yield

$$\pm R' > 0 \Rightarrow \{ \pm M' > 0, \quad \pm E' > 0, \quad \pm T' < 0 \}. \quad (22)$$

For **s5** in the region in which $E(r) \neq -1$, i.e. where R' does not change sign, differentiation of the exact solution for R with respect to r and investigation of the result as $\eta \rightarrow 2\pi$, and assuming $\mu \geq 0$, gives

$$\pm R' \geq 0 \Rightarrow \left\{ \pm M' \geq 0, \quad \left(E'M - \frac{2}{3}M'E + \frac{T'|E|^{5/3}}{3\pi} \right) \geq 0, \quad \pm T' \leq 0 \right\}. \quad (23)$$

Relations (22) and (23) are the Hellaby and Lake [6] no-shell-crossing conditions.

For values of r at which $E(r) = -1$, the condition $X \neq 0$ and finite implies that we must have

$$\frac{R'}{\sqrt{1+E}} \neq 0$$

and finite as $r \rightarrow r^*$ where $E(r^*) = -1$. Again, differentiation of the solution for R with respect to r gives

$$\lim_{r \rightarrow r^*} \frac{M'}{\sqrt{1+E}}, \quad \lim_{r \rightarrow r^*} \frac{T'}{\sqrt{1+E}}, \quad \lim_{r \rightarrow r^*} \frac{E'}{\sqrt{1+E}}$$

must be finite, and at least one must be non-zero. For a non-zero density at r^* , $\frac{M'}{\sqrt{1+E}}$ must be non-zero.

More generally, $\mu = 0$ in **s1** and **s2** from their definition. In **s3** and **s4**, μ is only zero if $M' = 0$; otherwise μ is finite for all r since $R'/M' \neq 0$. For **s5** it follows from above that $R'/M' \neq 0$, and hence the density is finite even where $E(r) = -1$. For **s6** the density vanishes if $B = 0$ and $A \neq 0$, otherwise it is finite and positive if and only if

$$\lim_{\pm} \frac{B}{A} \geq \frac{1}{\pi}.$$

We now consider the conditions for a centre ($R = 0$) to exist or be attached to a solution. Only comoving centres are possible and they may only join to solutions **s1** to **s5**. A list of the possibilities is given in table 1 [16].

Soln.	Behaviour of E , M and T	Kinematics	Example
(s1)	(i) $T \rightarrow 0$	$\Theta \equiv 0$	$T = -r$
	(ii) $\lim T'$ finite, nonzero	$\mu \equiv 0$	
(s2)	(i) $E \rightarrow 0$	$\Theta \rightarrow 3\epsilon\tau^{-1}$	$E = r^2$
	(ii) $\lim(E^{-1/2}E')$ finite, nonzero	$\mu \equiv 0$	$T = 0$
	(iii) $ET'/E' \rightarrow 0$		
(s3)	(i) $M \rightarrow 0$	$\Theta \rightarrow 2\epsilon\tau^{-1}$	$M = r^3$
	(ii) $\lim(M^{-2/3}M')$ finite, nonzero	$4\pi\mu \rightarrow \frac{2}{3}\tau^{-2}$	$T = 0$
	(iii) $MT'/M' \rightarrow 0$		
(s4)	(i) $E^{3/2}/M \rightarrow 0, M \rightarrow 0$	$\Theta \rightarrow 2\epsilon\tau^{-1}$	$E = r^3$
	(ii) $\lim(M^{-2/3}M')$ finite, nonzero	$4\pi\mu \rightarrow \frac{2}{3}\tau^{-2}$	$M = r^3$
	(iii) $\lim(MT'/M') = \lim(M^{1/3}E'/M') = 0$		$T = 0$
	(i) $E^{3/2}/M \rightarrow +\infty, E \rightarrow 0$	$\Theta \rightarrow 3\epsilon\tau^{-1}$	$E = r^2$
	(ii) $\lim(E^{-1/2}E')$ finite, nonzero	$4\pi\mu \rightarrow 0$	$M = r^4$
	(iii) $\lim(ET'/E') = \lim[E^{-1/2}M'E'^{-1}\log(E^{3/2}M^{-1})] = 0$		$T = 0$
(s5)	(i) $E^{3/2}/M \rightarrow \alpha > 0, M \rightarrow 0$	$\Theta \rightarrow 3\epsilon\alpha \sinh \eta / (\cosh \eta - 1)^2$	$E = r^2$
	(ii) $\lim(E^{-1}M')$ finite, nonzero	$4\pi\mu \rightarrow 3\alpha^2 / (\cosh \eta - 1)^3$	$M = r^3$
	(iii) $E^{-1}ME'/M' \rightarrow 2/3$ and $MT'/M' \rightarrow 0$		$T = 0$
	(i) $ E ^{3/2}/M \rightarrow 0, M \rightarrow 0$	$\Theta \rightarrow 2\epsilon\tau^{-1}$	$E = -r^3$
	(ii) $\lim(M^{-2/3}M')$ finite, nonzero	$4\pi\mu \rightarrow \frac{2}{3}\tau^{-2}$	$M = r^3$
	(iii) $\lim(MT'/M') = \lim(M^{1/3}E'/M') = 0$		$T = 0$
(s5)	(i) $ E ^{3/2}/M \rightarrow \alpha > 0, M \rightarrow 0$	$\Theta \rightarrow 3\epsilon\alpha \sin \eta / (1 - \cos \eta)^2$	$E = -r^2$
	(ii) $\lim(E^{-1}M')$ finite, nonzero	$4\pi\mu \rightarrow 3\alpha^2 / (1 - \cos \eta)^3$	$M = r^3$
	(iii) $E^{-1}ME'/M' \rightarrow 2/3$ and $MT'/M' \rightarrow 0$		$T = 0$

TABLE I. Central Behaviour

In the table, results labelled (i) arise from the behaviour as $R \rightarrow 0$; those labelled (ii) derive from the requirement that $X \neq 0$, and (iii) are a result of the shear vanishing. For each case, $M/R^3 \rightarrow \frac{4}{3}\pi\mu$, which is the Newtonian limit. Simple examples for which the centre lies at $r = 0$ are given in each case for illustration. The expansion rate at the centre,

$$\Theta = 2\frac{\dot{R}}{R} + \frac{\dot{X}}{X},$$

is listed for each solution. The central behaviours listed for **s4** and **s5** generalise previous results. An illustration of the calculations involved to derive the results in the table is given in [16].

V. MATCHING OF SOLUTIONS

In this section we reach the core of the paper. We examine matching across comoving space-like surfaces between solutions **s1** to **s6** to form composite models. The sign of R' cannot change across these interfaces since $E \neq -1$ on them - except on interfaces between **s5** and **s6**. However note that $R' \equiv 0$ in **s6**. Solutions **s1** may not be matched to any others because they are cosmological and it has $\dot{R} = 0$. Solution **s2** does not match to **s5** since ($E \rightarrow 0, M \rightarrow 0$) forces $R \rightarrow 0$ in **s5**. From the properties that characterise **s6** as an LTB model (see section 5.5), it follows that it only matches to **s5**. There remain just five physical types of junction.

A. (a) Matching **s2** to **s4**

The **s2** (interior) side of the junction is unconstrained by the matching. Approaching the junction from **s4**, $M \rightarrow 0$, $E > 0$ and $M' > 0$ (M' is continuous in **s4**) since $M > 0$ in **s4** and $M = 0$ in **s2**. Hence $R' > 0$ and R is increasing in the direction **s2** to **s4**. Denote the value of r at the junction by r^* . Then as $r \rightarrow r^*$, $\eta \rightarrow \infty$ because $M \rightarrow 0$ and $E > 0$ in the exact solutions (**s2**) and (**s4**). It follows that

$$X \rightarrow \frac{1}{\sqrt{1+E}} \left[\frac{M'}{E} \log \left(\frac{E^{3/2}}{M} \right) - T' E^{1/2} + \frac{E' \tau}{2E^{1/2}} \right]$$

and hence that X satisfies **R2** if $\lim_{(\mathbf{s4})} T', \lim_{(\mathbf{s4})} E'$ and $\lim_{(\mathbf{s4})} M' \ln M$ are finite and at least one is non-zero.

On the (**s4**) side, the density reduces to

$$4\pi\mu \rightarrow \left[\tau^2 \log \left(\frac{E^{2/3} T' \tau^2}{M} \right) - \frac{E^{3/2} T' \tau^2}{M'} + \frac{E^{1/2} E' \tau^3}{2M'} \right]^{-1}, \quad (24)$$

which goes to zero as $r \rightarrow r^*$. Note that $\mu = 0$ in (**s2**).

Since on the (**s4**) side $M \rightarrow 0$ as $r \rightarrow r^*$ and $M'M$ is finite, we must have $M' \rightarrow 0$ and hence

$$\frac{\dot{X}}{X} = \begin{cases} 0 & \text{if } E' \rightarrow 0, \\ \eta \left[\tau - \frac{2ET'}{E'} - \frac{2M'}{E^{1/2} E' \log M} \right]^{-1} & \text{otherwise.} \end{cases}$$

On the (**s2**) side

$$\frac{\dot{X}}{X} \rightarrow \frac{\epsilon}{\tau - 2ET'/E'},$$

and on both sides $\dot{R}/R \rightarrow \epsilon/\tau$. From this it follows that the shear remains finite on both sides of the junction.

B. (b) Matching **s3** to **s4**

The **s3** side is unconstrained by the matching. Approaching the junction from within the **s4** region, $E \rightarrow 0$ and $M > 0$. Since $E > 0$ in **s4**, we must have $E' > 0$ in a neighbourhood of the junction in the **s4** region. Also, from the explicit solution for **s4** we have

$$\sinh \eta - \eta = \tau E^{3/2} M^{-1} \rightarrow 0,$$

because $E \rightarrow 0$ and $M > 0$. Thus $\eta \approx (6\tau/M)^{1/3}E^{1/2}$ near the junction on the **(s4)** side and $E' > 0$, since $E > 0$ in the **(s4)** region. Hence R must increase in the direction **(s3)** to **(s4)**. On the **(s4)** side,

$$X \rightarrow M' \left(\frac{\tau^2}{6M^2} \right)^{1/3} - T' \left(\frac{4M}{3\tau} \right)^{1/3} + \frac{E'}{40} \left(\frac{(6\tau)^4}{M} \right)^{1/3}, \quad (25)$$

and so X is finite and non-zero provided

$$\lim_{(\mathbf{s4})} M', \lim_{(\mathbf{s4})} T' \text{ and } \lim_{(\mathbf{s4})} E' \text{ are finite and at least one is non-zero.}$$

On both sides of the interface, the density reduces to

$$4\pi\mu \rightarrow \begin{cases} 0 & \text{if } M' \rightarrow 0, \\ \left[\frac{3}{2}\tau^2 - \frac{3MT'\tau}{M'} - \frac{(6\tau)^{8/3}M^{1/3}E'}{160M'} \right]^{-1} & \text{otherwise,} \end{cases} \quad (26)$$

and on both sides,

$$\frac{\dot{X}}{X} \rightarrow \frac{\epsilon \left(T' + \frac{M'\tau}{M} + \left(\frac{243\tau^5}{250M^2} \right)^{1/3} E' \right)}{\left(\frac{3M'\tau^2}{2M} - 3T'\tau + \frac{E'}{10M^{2/3}} \left(\frac{9\tau^2}{2} \right)^{4/3} \right)}, \quad (27)$$

$$\frac{\dot{R}}{R} \rightarrow \frac{2\epsilon}{3\tau}. \quad (28)$$

Therefore μ and $\frac{\dot{X}}{X}$ and the shear are regular up to the junction.

C. (c) Matching s3 to s5

This junction is similar to (b). On approaching the junction in **(s5)**, $\eta \approx (6\tau M)^{1/3}|E|^{1/2} \rightarrow 0$. All the results in section 5.2 for the kinematics and metric components follow with **(s4)** replaced by **(s5)**.

In this case R must increase in the opposite sense to that in section 5.2, i.e., here R must increase in the direction **(s5)** to **(s3)**, as we will now show. Working in **(s5)**, at the junction $E = 0$, and since $E < 0$, it follows that $E' < 0$ in a neighbourhood of the junction. Also, near the junction the conditions (23) hold, so R' and M' have the same sign and since $M > 0$ and E may be as small as we please, R' and E' have the same sign. Since $E' < 0$, we have $R' < 0$, i.e., R increases in the direction from **(s5)** to **(s3)**. It is interesting to note that, since $\dot{R}/R \rightarrow \frac{2}{3}\epsilon\tau^{-1} > 0$ at a junction between **(s3)** and **(s5)**, the continuity of \dot{R}/R forces the existence of a finite region in **(s5)**, adjoining the junction, where the azimuthal expansion rate \dot{R}/R is positive even though all points in **(s5)** eventually satisfy $\dot{R} < 0$.

D. (d) Matching s4 to s5

Both sides are constrained by $E \rightarrow 0$ with $M > 0$ and the result is obtained by combining results from (b) and (c). In this case R must increase in the direction **(s5)** to **(s4)** by a similar argument to that given in (b).

E. (e) Matching s5 to s6

For solutions **(s6)**, $R' = 0$, and so they may only be matched, across a comoving surface, to solutions of the type **(s5)**, because $R' = 0$, $X \neq 0$ requires $E = -1$. This motivates a characterisation of **(s6)** within the family of LTB solutions by the conditions $M' = T' = 0$, $M > 0$ and $E = -1$.

At this junction the **(s6)** side is unconstrained by the matching. The observer area distance R may increase in either direction on approaching the junction from **(s5)**. Both metrics are regular on approach to the junction and it is less restrictive than the other four.

When the conditions for matching and for a centre are combined, four classes emerge.

VI. MODELS BY CLASS

A. Open models with one centre

By noting the sense in which R must increase at the interfaces (a) to (e) above, the only possible composite models are:

$$\begin{array}{ll} \mathcal{O}(\mathbf{s1})^+, & \mathcal{O}(\mathbf{s2})^+(\mathbf{s4})^+, \\ \mathcal{O}(\mathbf{s2})^+, & \mathcal{O}(\mathbf{s3})^+(\mathbf{s4})^+, \\ \mathcal{O}(\mathbf{s3})^+, & \mathcal{O}(\mathbf{s5})^+\mathcal{S}(\mathbf{s5})^+(\mathbf{s3})^+, \\ \mathcal{O}(\mathbf{s4})^+, & \mathcal{O}(\mathbf{s5})^+\mathcal{S}(\mathbf{s5})^+(\mathbf{s4})^+, \\ \mathcal{O}(\mathbf{s5})^+\mathcal{S}, & \mathcal{O}(\mathbf{s5})^+\mathcal{S}(\mathbf{s5})^+(\mathbf{s3})^+(\mathbf{s4})^+, \end{array}$$

where \mathcal{O} denotes a centre, and a superscript $+$ ($-$) implies that R increases (decreases) from left to right. Here \mathcal{S} is any combination of $(\mathbf{s5})^-$, $(\mathbf{s5})^+$ and $(\mathbf{s6})$. Note that open models can be constructed from collapsing solutions [e.g. $\mathcal{O}(\mathbf{s5})^+(\mathbf{s6})$]. Papapetrou [20] discussed a particular example of $\mathcal{O}(\mathbf{s5})^+(\mathbf{s3})$.

In the above construction, we have noted from (2) that on $t = \text{const}$, $d\chi = |dR|/\sqrt{1+E}$, where χ is radial proper distance. Hence by (21), if $E > \alpha > -1$ for all $\chi > \beta$ (α, β constants) then:

$$\chi \rightarrow \infty \text{ forces } R \rightarrow \infty \quad \text{if } \frac{dR}{d\chi} > 0, \quad (29)$$

$$\text{there is a finite value of } \chi > \beta \text{ for which } R = 0, \quad \text{if } \frac{dR}{d\chi} < 0. \quad (30)$$

However, if $E \rightarrow -1$ as $\chi \rightarrow \infty$, then neither of (29),(30) are necessary.

An example of $\mathcal{O}(\mathbf{s5})^+(\mathbf{s5})^-$ in the class of open models with one centre is

$$E = \begin{cases} -\frac{\sin^2 r}{\sin^2 r_0} [1 - e^{-2r_0}] & \text{for } 0 < r < r_0, \\ -1 + e^{-2r} & \text{for } r > r_0, \end{cases} \quad (31)$$

$$M = \begin{cases} \frac{\sin^3 r}{\sin^3 r_0} [M_\infty + e^{-r_0}] & \text{for } 0 < r < r_0, \\ M_\infty + e^{-r} & \text{for } r > r_0, \end{cases} \quad (32)$$

$$T = 0, \quad M_\infty > 0, \quad \pi < r_0 < 2\pi, \quad (33)$$

and

$$E = \begin{cases} -\frac{r^2}{r_0^2} [1 - e^{-2r_0}] & \text{for } 0 < r < r_0, \\ -1 + e^{-2r} & \text{for } r > r_0, \end{cases} \quad (34)$$

$$M = \begin{cases} \frac{r^3}{r_0^3} [M_\infty - e^{-r_0}] & \text{for } 0 < r < r_0, \\ M_\infty - e^{-r} & \text{for } r > r_0, \end{cases} \quad (35)$$

$$T = 0, \quad 0 < M_\infty < 2/3, \quad r_0 > 0, \quad (36)$$

is an example of $\mathcal{O}(\mathbf{s5})^+$. In each of (33) and (36), $R \rightarrow \text{const} > 0$ as $\chi \rightarrow \infty$. There are no spherically symmetric dust models with $R \rightarrow 0$ as $\chi \rightarrow \infty$ [by (29) and since, for the exact solutions, $R \rightarrow 0$ requires $E \rightarrow 0$].

B. Open models with no centre

By (30), to avoid a zero in R , a model with no centre must either be composed entirely of $(\mathbf{s6})$, or it must contain a section of $(\mathbf{s5})$, in order to allow (at least one) minimum in R . Then the possible matchings are evident:

$$\left. \begin{array}{l} (\mathbf{s3})^-(\mathbf{s5})^- \\ (\mathbf{s4})^-(\mathbf{s5})^- \\ (\mathbf{s4})^-(\mathbf{s3})^-(\mathbf{s5})^- \end{array} \right\} \mathcal{S} \left\{ \begin{array}{l} \mathcal{S} \\ (\mathbf{s5})^+(\mathbf{s3})^+ \\ (\mathbf{s5})^+(\mathbf{s4})^+ \\ (\mathbf{s5})^+(\mathbf{s3})^+(\mathbf{s4})^+ \end{array} \right.$$

Examples and a detailed analysis of such models are provided in [9]. In these models, due to the presence of collapsing solutions (s5), (s6), a centre does eventually form, but gravitational collapse will violate the regularity conditions in any case.

C. Closed models with two centres

These models must contain a region of (s5), since there must be (at least one) turning point in R . The models cannot contain a region of (s2), (s4) or (s2)(s4), since the region would either contain a centre and match to another solution, or would match to other solutions on both sides. Hence E would vanish on both sides, and since $E > 0$ throughout the domains of (s2) and (s4), E' could not have the same sign throughout, contrary to (22) [with (21)]. There can be no (s1) region in the closed model, since it does not match to any other solution. There can be no (s3) region in the model either, since R must increase in the direction (s5) \rightarrow (s3). Hence if (s3) contains a centre, it cannot match to (s5). Conversely, if (s3) does not contain a centre, it cannot match to (s5) on both sides, leaving the model open. This leaves just (s5) and (s6) to construct these models, and the possibilities are:

$$\mathcal{O}(\mathbf{s5})^+ \mathcal{S}(\mathbf{s5})^- \mathcal{O}$$

D. Closed models with no centre

Consider an SS dust model which has $R > 0$ in some range $0 \leq r \leq d$ (and at some t). This final possibility of composite models is obtained by identifying (matching) the surfaces $r = 0$ and $r = d$. Since $\Delta R = 0$, the model must be everywhere (s6) or else it must contain a region of (s5) [otherwise $\text{sign}(R')$ is constant in $0 \leq r \leq d$, which forces $R(0) \neq R(d)$]. No regions composed from the solutions (s1)-(s4) may be present, since they would be forced to match to (s5) on both sides. This would force R' to change sign in the region (since R must increase away from (s5) into these solutions) and this is not possible, by (21). Hence the models may only be constructed from (s5) and (s6), with the possibilities:

$$\mathcal{I} \mathcal{S} \mathcal{I}$$

where \mathcal{I} denotes the surfaces which are identified (at which the standard matching conditions must be satisfied, as we have described). The spatial sections of these models have the topology of a 3-torus. An example is provided by

$$E = \begin{cases} ar^2 - 1 & \text{for } 0 < r < \frac{1}{4}d, \\ a(r - \frac{1}{2}d)^2 - 1 & \text{for } \frac{1}{4}d < r < \frac{3}{4}d, \\ a(r - d)^2 - 1 & \text{for } \frac{3}{4}d < r < d, \end{cases} \quad (37)$$

$$M = \begin{cases} b + cr^2 & \text{for } 0 < r < \frac{1}{4}d, \\ b + \frac{1}{8}cd^2 - c(r - \frac{1}{2}d)^2 & \text{for } \frac{1}{4}d < r < \frac{3}{4}d, \\ b + c(r - d)^2 & \text{for } \frac{3}{4}d < r < d, \end{cases} \quad (38)$$

$$T = 0, \quad a \left(2b + \frac{1}{4}cd^2 \right) < \frac{4}{3}c, \quad (39)$$

where a, \dots, d are positive constants. Note that a closed model with no centre cannot be constructed from the homogeneous (Friedmann-Lemaître-Robertson-Walker) subclass of LTB (since the elliptic homogeneous solution has only one point with $E = -1$, at which R is maximum).

There are no further possible classes or composite models. There can be a number of different topologies constructed from these models but that involves different questions from those tackled here. Examples of models of types described in sections 6.1 to 6.3 are given in previous literature (see especially [9]).

VII. A BLACK HOLE IN AN EXPANDING UNIVERSE

In this section we present an example which serves to illustrate some points of significance in using matched LTB models in cosmology. First that combinations of different LTB models can provide realistic exact models of cosmologically interesting phenomena; second that the richness of the models is often overlooked; thirdly that even in simple cases, there are subtle issues that need to be watched (for instance an apparently useful model may be flawed) and fourthly the illustration itself has some intrinsic interest as a model of the formation of a black hole in an Einstein-de Sitter space-time which is different from the well known Oppenheimer-Snyder case.

In his treatise on inhomogeneous cosmological models [5] Krasiński gives the history of attempts to describe the formation of black holes in an expanding universe. The first successful attempt was by Barnes [21], who proved sufficient conditions for a black hole to form and he studied several examples of collapse in LTB coordinates. Subsequent work on similar lines was done by Demiański and Lasota [22] and Polnarev [23]. In 1978, Papapetrou published a model of a collapsing region in an Einstein-de Sitter spacetime. In his model the formation of the apparent horizon and the central singularity can easily be followed. In this model and in all others known to us, the singularity goes on accreting matter for an infinite LTB time, which limits their value in exact cosmology. In fact, although it is not widely known, this must happen irrespective of the precise nature of the collapsing part, unless it contains a vacuum region. The result follows from the matching conditions. If an **s5** region is matched to either an **s3** or **s4** one then, as we have seen, E must vanish at the boundary. It follows that when $\eta = 2\pi$, τ must be infinite on the boundary of the elliptic region (**s5**). Therefore, if the matter inside the elliptic region extends to its boundary then collapse to a singularity will take an infinite time to complete. We overcome this difficulty by arranging for the matter to extend only to some value of r within the elliptic (**s5**) region.

Although the end result in our model is an Einstein-Strauss vacuole the dynamics of the formation of the hole is different. In Oppenheimer-Snyder collapse, the mass gets trapped at the boundary first and at the centre last. In our case, the reverse happens and the horizon begins to form at the centre and spreads outwards, and the singularity is covered at all times.

A. The Model

To define the model we choose the following arbitrary functions:

$$T(r) = 0 \quad \forall r, \quad (40)$$

$$E(r) = \begin{cases} -\beta \left(\frac{r}{r_0}\right)^2 \left(1 - \frac{r}{r_0}\right)^2 & 0 \leq r \leq r_0, \\ 0 & r_0 < r, \end{cases} \quad (41)$$

$$M(r) = \begin{cases} \frac{1}{2}\alpha r^3 & r_0 < r, \\ \frac{1}{2}\alpha r_1^3 & r_1 \leq r \leq r_0, \\ \frac{1}{2}\alpha r^3 \left(\frac{r_1}{r_0}\right)^3 & r_0 < r. \end{cases} \quad (42)$$

The four arbitrary constants in the model are limited as follows:

1. $\alpha > 0$
2. $16 > \beta > 0$
3. $r_1 > \frac{1}{2}r_0$

The following properties of the model follow easily from well known results:

- a. From a formula of Barnes [21],

$$\frac{R'}{R} = \left(\frac{M'}{M} - \frac{E'}{E}\right) - \left(\frac{M'}{M} - \frac{3E'}{2E}\right) \frac{t\dot{R}}{R},$$

and the above, plus the appropriate exact solutions for R , it follows that $R' > 0$, as required to avoid a singularity in the metric.

- b. From condition (1) and $R' > 0$, the density is positive.

c. From (2) it follows that $(1 + E) > 0$, which ensures that r remains spacelike.

d. The junction conditions are satisfied for the (combined) metric, and it is non-singular except at $\eta = 0$ or 2π .

For $r > r_0$, the model is an Einstein-de Sitter universe with density $\rho = (6\pi t^2)^{-1}$. For $0 \leq r \leq r_0$, it represents an elliptic region which first expands and then collapses. In the region $r_1 < r \leq r_0$, it is vacuum, and for $0 \leq r \leq r_1$, it contains dust.

B. Collapse

Here we will discuss the dynamics of the elliptic region in some detail. It starts with a big bang at $t = 0$ and all shells expand until they reach their maximum surface area which occurs at $\eta = \pi$, i.e. at times given by

$$t = \frac{\pi M}{(-E)^{3/2}} \quad (43)$$

which is a monotonically increasing function of r . This means that shells with larger values of r reach their maximum later than those with smaller r . The shell bounding the matter, $r = r_1$, reaches its maximum at time

$$t = \frac{\pi \alpha r_0^3}{2[\beta^{1/2}(1 - \frac{r_1}{r_0})]^3}. \quad (44)$$

After reaching their maximum surface area, the shells collapse to a singularity at $R = 0$, which occurs when $\eta = 2\pi$, i.e. at time

$$t = \frac{\pi 2M}{(-E)^{3/2}}, \quad (45)$$

which depends on r . The point $r = 0$ is exceptional in that R vanishes for all t . However the behaviour of the density shows that after the universal singularity at $t = 0$, the point is non-singular until $\eta = 2\pi$. In LTB time, the singularity at $r = 0$ begins to form at time $t(0) = t_0$, where, taking limits,

$$t_0 = \frac{\pi \alpha r_0^3}{\beta^{3/2}}. \quad (46)$$

Note that the t_0 here is twice that found by Papapetrou [20]. The collapse process continues with the shells labelled r becoming singular at time

$$t = t_0 \left(1 - \frac{r}{r_0}\right)^{-3}. \quad (47)$$

The last shell at $r = r_1$ becomes singular at t_1 given by

$$t_1 = t_0 \left(1 - \frac{r_1}{r_0}\right)^{-3}, \quad (48)$$

after which the collapse is complete.

The mass in the singularity at any time $t_0 \leq t \leq t_1$ is given by

$$M = \frac{1}{2} \alpha r_0^3 \left[1 - \left(\frac{t_0}{t}\right)^{1/3}\right]^3 \quad (49)$$

Therefore at the beginning of the collapse, $M(t_0) = 0$ and at the end, $M(t_1) = \frac{1}{2} \alpha r_1^3$. Given the size of a spherically symmetric tophat region and the mass enclosed, we could determine α .

After time t_1 the solution is only defined for $r > r_1$, i.e., in the exterior vacuum and Einstein-de Sitter regions. The vacuum represents the Schwarzschild region in comoving coordinates.²

For $t > t_1$, the solution is equivalent to an Einstein-Strauss vacuole in an Einstein-de Sitter universe [24].

²Singularities continue to form for $r > r_1$ at times

$$t = t_0 \left(\frac{r_1}{r_0}\right)^3 \left(1 - \frac{r}{r_0}\right)^{-3},$$

C. Horizons

As usual in LTB spacetimes, we use the apparent horizon as diagnostic for the existence of an event horizon and therefore a black hole. The apparent horizon is given by [21]³

$$R(r, t) = 2M(r), \quad \dot{R} < 0. \quad (50)$$

This formula can be put in the alternative forms

$$\dot{R} = -(1 - E)^{1/2} \quad (51)$$

and

$$\sin(\eta/2) = (-E)^{1/2}, \quad \pi < \eta \leq 2\pi. \quad (52)$$

To simplify the notation, we define $w := \frac{r}{r_0}$. Then (52) gives

$$\sin(\eta/2) = \beta^{1/2} w(1 - w), \quad \pi < \eta \leq 2\pi, \quad (53)$$

and from the definition of η in solution (s5) and the definitions of the arbitrary functions,

$$t_{AH}(r) = \frac{t_0(\eta - \sin \eta)}{2\pi(1 - w)^3}, \quad 0 \leq w \leq w_1, \quad (54)$$

$$t_{AH}(r) = \left(\frac{t_0(\eta - \sin \eta)}{2\pi(1 - w)^3} \right) \left(\frac{w_1}{w} \right), \quad w_1 \leq w \leq 1, \quad (55)$$

where $w_1 := r_1/r_0$ and η is determined by (53). Equations (54) and (55) together give the equation of the apparent horizon. The areal radius of the apparent horizon at coordinate r is given by

$$R_{AH} = \alpha r^3, \quad 0 \leq r \leq r_1, \quad (56)$$

$$R_{AH} = \alpha r_1^3, \quad r_1 \leq r \leq r_0, \quad (57)$$

where we have used the definition of M for the appropriate range of r and equation (50).

From (53) if $w = 0$ then $\eta = 2\pi$ and so (54) gives $t_{AH} = t_0$. Thus at $r = 0$ the singularity and the apparent horizon form together. When this happens, the singularity may be naked [26–28], but we will not discuss that here. We will concentrate instead on the formation of the black hole and assume that $0 < w \leq 1$.

We denote the time at which the singularity forms at r by $t_s(r)$, which is given by (47), i.e.,

$$t_s(r) = t_0(1 - w)^{-3}. \quad (58)$$

Then for $0 \leq r \leq r_1$, i.e., inside the collapsing dust sphere, we obtain from (54)

$$t_s(r) - t_{AH}(r) = \frac{t_0(2\pi - \eta + \sin \eta)}{2\pi(1 - w)^3}, \quad (59)$$

and for $r_1 \leq r \leq r_0$, i.e., in the surrounding vacuum,

$$t_s(r) - t_{AH}(r) = \frac{t_0(2\pi - \eta + \sin \eta)}{2\pi(1 - w)^3} \left(\frac{w_1}{w} \right)^3, \quad (60)$$

where η is given by equation (53). Given the range of η , $t_s - t_{AH} > 0$ always. So the apparent horizon forms first and the singularity is not naked. This agrees with a result of Joshi [29]. The horizons starts to form at the centre and spreads outwards with time. This is different from the behaviour of the Oppenheimer-Snyder solution for which the boundary and the mass get trapped first and the centre last.

In the limiting case where $w \rightarrow 1$, the inner collapsing sphere extends to the Einstein-de Sitter exterior, so that there is no vacuum region. This is the case considered by Papapetrou and analysis of the limits confirms his result. Both t_s and t_{AH} tend to ∞ . However the difference between them remains finite and t_s remains greater than t_{AH} .

and are to be interpreted as the arrival of successive shells of *test particles* which label the coordinates of the vacuum in this gauge. This process continues until the shell $r = r_0$ arrives at time $t = \infty$.

³This is the apparent horizon bounding trapped surfaces associated with the collapse, which begins at $\eta = \pi$. There is also an apparent horizon related to the initial expansion which all particles must cross during their expansion phase from the big bang *white hole*. Gautreau and Cohen [25] call this a boundary of expelled surfaces.

VIII. CONCLUSION

Here a set of spherically symmetric inhomogeneous dust models has been provided which can be used to construct cosmological solutions to Einstein's field equations for a range of astrophysical situations including voids and black holes. A section of the book by Krasinski [5] is devoted to applications of these models in cosmology. In setting out the models new results have been obtained which fill gaps in the literature. In particular we have demonstrated the matching of the Kantowski-Sachs solutions and derived a new solution where the spatial sections have a torus topology. Also the particular regularity conditions we use lead to a restricted but, we would argue, cosmologically more useful set of matched solutions. The categorisation we provide of the allowed cases with centres is useful for the construction of models.

The value of these matched solutions is demonstrated in the final example of black hole formation which illustrates very clearly why it is important to consider details when using inhomogeneous models. We have shown that using matched LTB solutions a model can be constructed to describe collapse to a black hole in an Einstein de Sitter background but it must contain a vacuum region if the singularity is not to continue to accrete matter for an infinite LTB time.

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